Toric Prevarieties and Subtorus Actions

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Abstract. Dropping separatedness in the definition of a toric variety, one obtains the more general notion of a toric prevariety. Toric prevarieties occur as ambient spaces in algebraic geometry and moreover they appear naturally as intermediate steps in quotient constructions. We first provide a complete description of the category of toric prevarieties in terms of convex–geometrical data, so–called systems of fans. In a second part, we consider actions of subtori H of the big torus of a toric prevariety X and investigate quotients for such actions. Using our language of systems of fans, we characterize existence of good prequotients for the action of H on X. Moreover, we show by means of an algorithmic construction that there always exists a toric prequotient for the action of H on X, that means an H–invariant toric morphism p from X to a toric prevariety Y such that every H–invariant toric morphism from X to a toric prevariety factors through p. Finally, generalizing a result of D. Cox, we prove that every toric prevariety X occurs as the image of a categorical prequotient of an open toric subvariety of some $\mathbb{C}^s$.

Introduction

A *toric prevariety* is a normal prevariety, i.e. possibly non–separated, together with an effective regular action of an algebraic torus that has a dense orbit. This notion occurs for the first time in an article by J. Włodarczyk in 1991 (see [13]) where he shows that toric prevarieties are in fact universal ambient spaces in algebraic geometry. More precisely, he proves: *Every normal variety over an algebraically closed field admits a closed embedding into a toric prevariety.*

In classical algebraic geometry, all the varieties considered were quasi–projective; so the ambient spaces were finite dimensional vector spaces or projective spaces. But when in the 1940s the abstract concept of an algebraic variety via glueing affine charts was introduced, the resulting class of objects turned out to be much larger than those fitting into the frame of the classical ambient spaces. It therefore came as a surprise that the class of toric prevarieties is indeed so large that they contain any given normal complex variety as a closed subvariety.

However, non–separated toric prevarieties have so far hardly been studied. In the first part of this article our aim is to provide a complete description of complex toric prevarieties and their morphisms in terms of convex geometry. Generalizing the notion of a fan, we introduce the concept of a *system of fans* in a lattice and obtain an equivalence of the category of systems of fans and the category of toric prevarieties (see Theorem 3.6).

Our description makes the category of toric prevarieties accessible for explicit calculations. A first application is an algorithmic construction of a *toric separation*: For every toric prevariety X, we obtain a toric morphism from X to a toric variety Y that is universal with respect to toric morphisms from X to toric varieties (see Theorem 4.1). This result can serve as a general tool for passing from the non–separated setting to varieties. Further applications

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of the convex–geometrical language are given in subsequent articles. For example, it is used in [7] to prove a refined version of Włodarczyk's embedding theorem.

The second part of this article is devoted to quotient constructions. Frequently it is useful to decompose a quotient construction into a non–separated first step followed by a separation process. For example, A. Białynicki–Birula uses non–separated quotient spaces as a tool to prove in [3] that a normal variety contains only finitely many maximal subsets admitting a good quotient with respect to a given reductive group action.

In the toric setting it is natural to consider subtorus actions. Here one faces the following problem: Given a toric variety X and a subtorus H of the big torus of X; when does there exist a suitable quotient for the action of H on X? This quotient problem has been studied by various authors: In [9] GIT–quotients for subtorus actions on projective toric varieties were investigated. More generally, J. Święcicka ([12]) and H. Hamm ([8]) asked for existence of arbitrary good quotients. In section 6 we treat their questions in the framework of non–separated toric prevarieties. In analogy to the corresponding concept for the separated case, we define the notion of a *good prequotient*. We characterize in terms of systems of fans when a good prequotient for the action of a subtorus H on a toric prevariety X exists (see Theorem 6.7).

The characterizations given in the above–mentioned results show that good quotients and prequotients only exist under quite special circumstances. So it is natural to ask for more general notions. In [1], the notion of a *toric quotient* for a subtorus action on a toric variety was introduced and it was proved that such a quotient always exists. The analogous notion in the context of toric prevarieties is the *toric prequotient*, i.e., an H–invariant toric morphism p from X to a toric prevariety Y such that every H–invariant toric morphism from X factors uniquely through p. In Section 7 we prove by means of an explicit algorithm (see Theorem 7.5) that toric prequotients always exist. Determining first the toric prequotient of a subtorus action on a toric variety and then performing toric separation splits the calculation of the toric quotient into two steps (see Remark 7.8). This decomposition gives for example insight into obstructions to the existence of categorical quotients (see [2]).

An application of our theory of quotients is given in the last section: we represent an arbitrary toric prevariety as a quotient space of an open toric subvariety of some $\mathbb{C}^n$ by the action of a subtorus of $(\mathbb{C}^*)^n$ (see Corollary 8.3). This result generalizes a similar statement on toric varieties due to D. Cox (see [5]). In contrast to Cox's construction, our quotient map is not necessarily a good prequotient. However, it is universal in the category of algebraic prevarieties (see Proposition 8.2). A toric prevariety X with big torus T occurs as the image of a good prequotient of an open subvariety of some $\mathbb{C}^s$ if and only if the intersection of any two maximal affine open T–stable subspaces of X is affine (see Theorem 8.8).

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1 Toric Prevarieties

Let X be a complex algebraic prevariety, i.e., a connected complex ringed space that is obtained by glueing finitely many complex affine varieties along open subspaces. Recall that X is separated if and only if it is Hausdorff with respect to the complex topology.

By definition a prevariety X is normal if it is irreducible and all its local rings are integrally closed domains, or equivalently, all its affine charts are normal. As in the case of varieties, a normalization of a given prevariety X is obtained by glueing normalizations of affine charts of X.

1.1 Definition. A *toric prevariety* is a normal complex prevariety X together with an effective regular action of an algebraic torus T having an open orbit.

For a toric prevariety X and T as in 1.1, we refer to T as the *acting torus* of X. Moreover, we fix a *base point* x_0 in the open orbit of X. Note that a toric prevariety is a toric variety in the usual sense (see e.g. [6]) if and only if it is separated.

1.2 Example. The complex line, endowed with the $\mathbb{C}^*$–action $t \cdot z := tz$, is a toric variety. Glueing two disjoint copies of $\mathbb{C}$ along the open orbit $\mathbb{C}^* \subset \mathbb{C}$ yields a non-separated toric prevariety X. As a base point we choose $x_0 := 1 \in \mathbb{C}^* \subset X$. $\diamondsuit$

1.3 Proposition. *Every toric prevariety X admits a finite covering of open affine subspaces that are stable by the acting torus T of X.*

Proof. According to [3], Theorem 1, there are only finitely many maximal separated open subspaces U_i, $i \in I$, of X. Since X is a noetherian topological space, it is covered by the U_i.

By Sumihiro's Theorem (see [11]) we only have to show that each U_i is T–stable. This is done as follows: Every $t \in T$ permutes the sets U_i. Hence the elements of T permute also the complements $A_i := X \setminus U_i$. Consequently, for a given $i \in I$ we have

$$T = \bigcup_{j \in I} \mathrm{Tran}_T(A_i, A_j),$$

where $\mathrm{Tran}_T(A_i, A_j)$ denotes the closed set $\{t \in T;\ t \cdot A_i \subset A_j\}$. Since T is irreducible, there is a $j_0 \in I$ such that $T = \mathrm{Tran}_T(A_i, A_{j_0})$. Note that $A_i = e_T \cdot A_i \subset A_{j_0}$. Thus maximality of U_{j_0} implies $i = j_0$ which yields $T \cdot U_i = U_i$. $\square$

Note that the arguments of the above proof yield that any G–prevariety, where G is a connected algebraic group, can be covered by G–stable separated open subspaces. For disconnected G this statement is false (see Example 1.6).

Now assume that X is a toric prevariety with acting torus T. Using the theory of toric varieties we can conclude from Proposition 1.3 that the set $\mathrm{Orb}(X)$ of all T–orbits of X is finite.

1.4 Remark. For every T–orbit B of X there exists a unique T-stable open affine subspace X_B of X such that B is a closed subset of X_B. Moreover, we have

$$X_B = \bigcup_{B' \in \mathrm{Orb}(X);\ B \subset \overline{B'}} B'. \qquad \diamondsuit$$

The morphisms in the category of toric prevarieties are defined similarly as in the separated case: Let $f\colon X \to X'$ be a regular map of toric prevarieties X and X' with base points x_0 and x_0' respectively.

1.5 Definition. The map f is called a *toric morphism* if $f(x_0) = x_0'$ and there is a homomorphism $\varphi\colon T \to T'$ of the acting tori of X and X' such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ holds for all $(t, x) \in T \times X$.

1.6 Example. For the toric prevariety X of Example 1.2 let 0_1 and 0_2 denote the two fixed points of T. Then $f|_{\mathbb{C}^*} := \mathrm{id}_{\mathbb{C}^*}$, $f(0_1) := 0_2$ and $f(0_2) := 0_1$ defines a toric automorphism of X of order 2.

1.7 Lemma. *Let $f\colon X \to X'$ be a toric morphism of toric prevarieties and let $B \subset X$, $B' \subset X'$ be orbits of T and T' respectively. Then we have $f(X_B) \subset X'_{B'}$ if and only if $f(B) \subset X'_{B'}$.*

Proof. Suppose that $f(B) \subset X'_{B'}$ holds. Then Remark 1.4 yields $B' \subset \overline{T' \cdot f(B)}$. Now consider a T–orbit B_1 with $B \subset \overline{B_1}$. Then $f(B) \subset \overline{f(B_1)}$ and hence B' is contained in the closure of the T'-orbit $T' \cdot f(B_1)$. That means that $f(B_1) \subset X'_{B'}$. Thus Remark 1.4 implies $f(B) \subset X'_{B'}$. $\square$

2 Systems of Fans

In this section we introduce the notion of a system of fans in a lattice and associate to every system of fans a toric prevariety. Our construction is a generalization of the basic construction in the theory of toric varieties. First we have to fix some notation:

By a lattice we mean a free $\mathbb{Z}$–module of finite rank. For a given lattice N let $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N$ denote its associated real vector space. Moreover, for a homomorphism $F \colon N \to N'$ of lattices, denote by $F_{\mathbb{R}}$ its extension to the real vector spaces associated to N and N'.

In the sequel let N be a lattice. When we speak of a cone in N we always think of a (not necessarily strictly) convex rational polyhedral cone in $N_{\mathbb{R}}$. For a cone σ in N we denote by σ° the relative interior of σ and if τ is a face of σ, then we write $\tau \prec \sigma$.

As usual, we call a finite set Δ of strictly convex cones in N a *fan* in N if any two cones of Δ intersect in a common face and if $\sigma \in \Delta$ implies that also every face of σ lies in Δ. If Δ' is a subfan of a fan Δ we will write $\Delta' \prec \Delta$. A fan Δ is called *irreducible* if it consists of all the faces of a cone σ.

2.1 Definition. Let I be a finite index set. A collection $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ of fans in N is called a *system of fans* if the following properties are satisfied for all $i, j, k \in I$:

i) $\Delta_{ij} = \Delta_{ji}$,

ii) $\Delta_{ij} \cap \Delta_{jk} \prec \Delta_{ik}$.

Note that in particular, $\Delta_{ij} \prec \Delta_{ii} \cap \Delta_{jj}$ for all $i, j \in I$.

2.2 Examples. i) Every fan Δ in N can be considered as a system of fans $\mathcal{S} = (\Delta)$ with just one element.

ii) Let $\sigma_1, \ldots, \sigma_r$ denote the maximal cones of a fan Δ and set $I := \{1, \ldots, r\}$. Let Δ_{ii} denote the fan of faces of σ_i and define $\Delta_{ij} := \Delta_{ii} \cap \Delta_{jj}$. Then $\mathcal{S} := (\Delta_{ij})_{i,j \in I}$ is a system of irreducible fans.

iii) If $\Delta_1, \ldots, \Delta_r$ are fans in a lattice N, then $\Delta_{ii} := \Delta_i$ and $\Delta_{ij} := \{\{0\}\}$ defines a system of fans in N.

iv) For a given collection $\sigma_1, \ldots, \sigma_r$ of strictly convex cones in a lattice N, set $I := \{1, \ldots, r\}$. Let Δ_{ii} denote the fan of faces of σ_i and define Δ_{ij} to be the fan of all common proper faces of σ_i and σ_j. Then $\mathcal{S} := (\Delta_{ij})_{i,j \in I}$ is a system of fans. $\diamondsuit$

In the sequel let $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ be a given system of fans in N. There is an algebraic torus having N as its lattice of one parameter subgroups, namely $T := \operatorname{Hom}(N^{\vee}, \mathbb{C}^*)$, where $N^{\vee} := \operatorname{Hom}(N, \mathbb{Z})$ denotes the dual module of N. We associate to $\mathcal{S}$ a toric prevariety $X_{\mathcal{S}}$ with acting torus T as follows:

For each index $i \in I$ let $X_i := X_{\Delta_{ii}}$ denote the toric variety corresponding to the fan Δ_{ii} (see e.g. [6]). For any two indices $i \neq j$ let X_{ij} and X_{ji} be the open toric subvarieties of X_i and X_j corresponding to the subfan Δ_{ij}.

The lattice homomorphism id_N defines toric isomorphisms $f_{ji} \colon X_{ij} \to X_{ji}$. Note that Property 2.1 ii) yields $f_{kj} \circ f_{ji} = f_{ki}$ on the intersections $X_{ij} \cap X_{ik}$. Define $X_{\mathcal{S}}$ to be the

T–equivariant glueing of the toric varieties X_i by the glueing maps f_{ij}. By construction, $X_{\mathcal{S}}$ is a toric prevariety.

2.3 Example. Let $I := \{1, 2\}$ and let $\Delta_{11} = \Delta_{22} := \{\{0\}, \sigma\}$ be the fan of faces of $\sigma := \mathbb{R}_{\geq 0}$. Setting $\Delta_{12} := \Delta_{21} := \{\{0\}\}$ we obtain a system of fans $\mathcal{S}$ in $\mathbb{Z}$. The associated toric prevariety $X_{\mathcal{S}}$ is the complex line with zero doubled (see Example 1.2). $\diamondsuit$

Apparently, different systems $\mathcal{S}$ and $\mathcal{S}'$ of fans can lead to the same toric prevariety X, since there may be various possibilites for choosing separated toric charts covering the prevariety X.

2.4 Example. Let $N := \mathbb{Z}$ and $\sigma := \mathbb{R}_{\geq 0}$. The fan $\Delta := \{\sigma, -\sigma, \{0\}\}$ gives rise to the toric variety $X_\Delta = \mathbb{P}_1$. Let $I := \{1, 2\}$ and set

$$\Delta_{11} := \Delta_{22} := \Delta, \qquad \Delta_{12} := \Delta_{21} := \{\{0\}\}.$$

Then the resulting system of fans $\mathcal{S} := (\Delta_{ij})_{i,j \in I}$ defines the toric prevariety X that is obtained from glueing two copies of $\mathbb{P}_1$ along $\mathbb{C}^*$. If we set $I' := \{1, 2, 3, 4\}$,

$$\Delta'_{11} := \Delta'_{22} := \{\{0\}, \sigma\}, \qquad \Delta'_{33} := \Delta'_{44} := \{\{0\}, -\sigma\},$$

and $\Delta'_{ij} := \{\{0\}\}$ for all $i \neq j$, then we arrive at a system $\mathcal{S}'$ of fans defining the same toric prevariety X as above. But now the fans of the system are irreducible and correspond to affine charts of X. $\diamondsuit$

A given toric prevariety has two distinguished systems of charts, namely the covering by maximal T–stable separated charts and the covering by maximal T–stable affine charts. The latter one corresponds to systems $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ of fans where every Δ_{ii} is irreducible. Such a system will be called *affine*.

For the description of the orbit structure of $X_{\mathcal{S}}$ the following observation will turn out to be useful: The system of fans $\mathcal{S}$ naturally induces an equivalence relation on the set $\mathfrak{F}(\mathcal{S}) := \{(\sigma, i);\ i \in I, \sigma \in \Delta_{ii}\}$ of labelled cones, namely

$$(\sigma, i) \sim (\sigma, j) \quad \Longleftrightarrow \quad \sigma \in \Delta_{ij}\,.$$

We call this equivalence relation the *glueing relation* of $\mathcal{S}$, and we denote the set of equivalence classes by $\Omega := \Omega(\mathcal{S})$. The equivalence class of an element $(\sigma, i) \in \mathfrak{F}(\mathcal{S})$ is denoted by $[\sigma, i]$.

2.5 Remark. The glueing relation satisfies the following conditions:

i) $(0, i) \sim (0, j)$ for all i, j,

ii) $(\sigma, i) \sim (\tau, j)$ implies $\sigma = \tau$,

iii) $(\tau, i) \sim (\tau, j)$ implies $(\tau', i) \sim (\tau', j)$ for every $\tau' \prec \tau$. $\diamondsuit$

As a converse of Remark 2.5, we can recover $\mathcal{S}$ from its glueing relation: Let S denote a finite set of cones in N, let I be a finite index set and suppose that $\mathfrak{F}$ is a subset of $S \times I$ where for every i the set $\Delta_{ii} := \mathfrak{F} \cap (S \times \{i\})$ forms a fan.

2.6 Remark. If $\sim$ is an equivalence relation on $\mathfrak{F}$ satisfying the conditions 2.5 i)–iv), then we obtain a system of fans by setting

$$\Delta_{ij} := \{\tau \in \Delta_{ii} \cap \Delta_{jj};\ (\tau, i) \sim (\tau, j)\}\,. \qquad \diamondsuit$$

Let us now return to the toric prevariety $X_{\mathcal{S}}$ obtained from glueing the toric varieties $X_i = X_{\Delta_{ii}}$. Recall that the T–orbits of X_i are in 1–1–correspondence with the cones in Δ_{ii}. For every $\sigma \in \Delta_{ii}$, there is even a distinguished point $x_{(\sigma,i)}$ in the corresponding T–orbit in X_i (see e.g. [6], p.28).

In the toric prevariety $X_{\mathcal{S}}$ a point $x_{(\sigma,i)} \in X_i$ is identified with $x_{(\tau,j)} \in X_j$ if and only if $x_{(\sigma,i)} \in X_{ij}$ and $x_{(\tau,j)} \in X_{ji}$ and $\sigma = \tau$, or equivalently if $(\sigma, i) \sim (\tau, j)$. So a distinguished point $x_{(\sigma,i)} \in X_i$ defines a *distinguished point* in $X_{\mathcal{S}}$ which depends only on the equivalence class $[\sigma, i]$ of (σ, i) in $\Omega(\mathcal{S})$ and is denoted by $x_{[\sigma,i]}$.

2.7 Remark. The assignment $[\sigma, i] \mapsto T \cdot x_{[\sigma,i]}$ defines a bijection from $\Omega(\mathcal{S})$ to the set of T-orbits of the toric prevariety $X_{\mathcal{S}}$. $\diamondsuit$

The point $x_0 := x_{[\{0\},i]}$ corresponding to the open T-orbit will be considered as the *base point* of $X_{\mathcal{S}}$. For a distinguished point $x_{[\sigma,i]}$ of $X_{\mathcal{S}}$ we define $X_{[\sigma,i]}$ to be the open affine T–stable subspace of $X_{\mathcal{S}}$ that contains $T \cdot x_{[\sigma,i]}$ as closed subset (see Remark 1.4).

By Property 2.5 iv), the face relation induces a partial ordering on the set $\Omega(S)$, namely $[\tau, j] \prec [\sigma, i]$ if τ is a face of σ and $[\tau, i] = [\tau, j]$. This partial ordering reflects the behaviour of orbit closures in $X_{\mathcal{S}}$:

2.8 Lemma. *A point $x_{[\sigma,i]}$ lies in the closure of the orbit $T \cdot x_{[\tau,j]}$ if and only if $[\tau, j] \prec [\sigma, i]$. In particular, one has*

$$X_{[\sigma,i]} = \bigcup_{[\tau,j]\prec[\sigma,i]} T \cdot x_{[\tau,j]} = \bigcup_{[\tau,j]\prec[\sigma,i]} X_{[\tau,j]}.$$

Proof. Assume $[\tau, j] \prec [\sigma, i]$. By definition of the partial ordering "$\prec$", this means $\tau \prec \sigma$ and $[\tau, j] = [\tau, i]$. This implies $x_{(\sigma,i)} \in \overline{T \cdot x_{(\tau,i)}} \subset X_i$. Hence $x_{[\sigma,i]}$ lies in the closure of $T \cdot x_{[\tau,j]}$.

Now, let $x_{[\sigma,i]} \in \overline{T \cdot x_{[\tau,j]}}$. Consider the T–stable separated neighbourhood X_i of $x_{[\sigma,i]}$. Since $X \setminus X_i$ is closed, we have $x_{[\tau,j]} \in X_i$, i.e., $[\tau, j] = [\tau, i]$. Now the theory of toric varieties tells us that in X_i we have $\tau \prec \sigma$. $\square$

Together with the corresponding statement on affine toric varieties, Lemma 1.7 implies the following

2.9 Remark. Let $f: X_{\mathcal{S}} \to X_{\mathcal{S}'}$ be a toric morphism. Then f maps distinguished points to distinguished points. $\diamondsuit$

3 Toric Morphisms and Maps of Systems of Fans

We first introduce the concept of a map of systems of fans and then show that $\mathcal{S} \mapsto X_{\mathcal{S}}$ is an equivalence of categories. Let $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ and $\mathcal{S}' = (\Delta'_{ij})_{i,j \in I'}$ denote systems of fans in lattices N and N' respectively.

3.1 Definition. A *map of systems of fans* from $\mathcal{S}$ to $\mathcal{S}'$ is a pair $(F, \mathfrak{f})$, where $F: N \to N'$ is a lattice homomorphism and $\mathfrak{f}: \Omega(S) \to \Omega(S')$ is a map with the following properties:

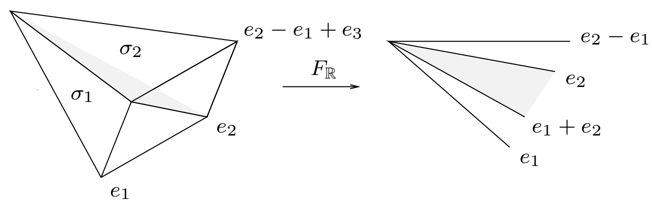
i) If $[\tau, j] \prec [\sigma, i]$ then $\mathfrak{f}([\tau, j]) \prec \mathfrak{f}([\sigma, i])$, i.e., $\mathfrak{f}$ is order preserving.

ii) If $\mathfrak{f}([\sigma, i]) = [\sigma', i']$ then $F_{\mathbb{R}}(\sigma^{\circ}) \subset (\sigma')^{\circ}$.

3.2 Remark. Assume that $\mathcal{S}' = (\Delta')$ is a single fan in N' and $F: N \to N'$ is a lattice homomorphism such that $F_{\mathbb{R}}$ maps the cones of $\mathcal{S}$ into cones of Δ'. Then there is a unique map $\mathfrak{f}: \Omega(\mathcal{S}) \to \Omega(\mathcal{S}')$ such that $(F, \mathfrak{f})$ is a map of the systems of fans $\mathcal{S}$ and $\mathcal{S}'$. $\diamondsuit$

On the other hand, if $\mathcal{S}'$ is arbitrary but $\mathcal{S} = (\Delta)$ is a single fan and $F\colon N \to N'$ is a lattice homomorphism such that $F_{\mathbb{R}}$ maps cones of $\mathcal{S}$ into cones of $\mathcal{S}'$, then there need not exist a map $(F, \mathfrak{f})$ of the systems of fans $\mathcal{S}$ and $\mathcal{S}'$, even if the glueing relation of $\mathcal{S}'$ is maximal:

3.3 Example. Let Δ be the fan in $\mathbb{Z}^3$ having $\sigma_1 := \mathrm{cone}(e_1, e_2, e_1 + e_2 + e_3)$ and $\sigma_2 := \mathrm{cone}(e_2, e_1 + e_2 + e_3, e_2 - e_1 + e_3)$ as its maximal cones. Let $F\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ denote the projection given by $(x, y, z) \mapsto (x, y)$.



Let $I' := \{1, 2\}$ and let Δ_{ii} be the fan of faces of the cones $\tau_i := F_{\mathbb{R}}(\sigma_i)$ in $\mathbb{Z}^2$. Let $\mathcal{S}'$ denote the system of fans obtained obtained from the Δ_{ii} by adding $\Delta_{12} := 0$. Then there is no map of systems of fans $(F, \mathfrak{f})$ from $\mathcal{S}$ to $\mathcal{S}'$. $\diamondsuit$

To obtain a functor from the category of systems of fans to the category of toric prevarieties we now define the assignement on the level of morphisms. Let $X := X_{\mathcal{S}}$ and $X' := X_{\mathcal{S}'}$ be the respective toric prevarieties arising from $\mathcal{S}$ and $\mathcal{S}'$ and let $(F, \mathfrak{f})$ be a map of the systems of fans $\mathcal{S}$ and $\mathcal{S}'$.

We construct a toric morphism $f\colon X \to X'$ as follows. For a given $i \in I$ and $\sigma \in \Delta_{ii}$, set $[\sigma', i'] := \mathfrak{f}([\sigma, i])$. Then $F_{\mathbb{R}}(\sigma) \subset \sigma'$, so F defines a toric morphism $f_{[\sigma,i]}$ from the affine toric variety $X_{[\sigma,i]}$ to $X'_{[\sigma',i']}$. By condition 3.1 i) we obtain

$$f_{[\sigma,i]}|_{X_{[\sigma,i]} \cap X_{[\tau,j]}} = f_{[\tau,j]}|_{X_{[\sigma,i]} \cap X_{[\tau,j]}}$$

for every $[\tau, j] \in \Omega(\mathcal{S})$. Consequently the regular maps $f_{[\sigma,i]}$ glue together to a toric morphism $f\colon X \to X'$.

The geometric meaning of the map $\mathfrak{f}\colon \Omega(\mathcal{S}) \to \Omega(\mathcal{S}')$ is to prescribe the values of the distinguished points for the toric morphism f associated to a map $(F, \mathfrak{f})$ of the systems $\mathcal{S}$ and $\mathcal{S}'$ of fans:

3.4 Lemma. *For every* $[\sigma, i] \in \Omega(\mathcal{S})$ *we have* $f(x_{[\sigma,i]}) = x'_{\mathfrak{f}([\sigma,i])}$.

Proof. Let $[\sigma, i] \in \Omega(\mathcal{S})$ and let $[\sigma', i'] := \mathfrak{f}([\sigma, i])$. Consider the toric morphism $f_i := f_{[\sigma,i]}\colon X_{[\sigma,i]} \to X'_{[\sigma',i']}$. Since $F_{\mathbb{R}}(\sigma^\circ)$ is contained in $(\sigma')^\circ$, we have

$$f_i(x_{(\sigma,i)}) = f_i\left(\lim_{t\to 0} \lambda_v(t) \cdot x_{(\{0\},i)}\right) = \lim_{t\to 0} f_i(\lambda_v(t) \cdot x_{(\{0\},i)}) = \lim_{t\to 0} \lambda_{F(v)}(t) \cdot x_{(\{0\},i)} = x_{(\sigma',i')}$$

where v is any lattice point in σ° and $\lambda_v\colon \mathbb{C}^* \to T$ denotes the one–parameter–subgroup of the acting torus T of $X_{[\sigma,i]}$ defined by v. This yields the claim. $\square$

Denote by $\varphi\colon T \to T'$ the homomorphism of acting tori associated to the toric morphism f. Then we obtain the following description of the fibers of f.

3.5 Proposition. *Fibre Formula. For every distinguished point* $x_{[\sigma',i']} \in X_{\mathcal{S}'}$ *we have*

$$f^{-1}(x_{[\sigma',i']}) \ = \bigcup_{[\sigma,i] \in \mathfrak{f}^{-1}([\sigma',i'])} \varphi^{-1}(T'_{x_{[\sigma',i']}}) \cdot x_{[\sigma,i]}.$$

Proof. The inclusion "$\supset$" follows from Lemma 3.4. In order to check "$\subset$", let $x \in f^{-1}(x_{[\sigma',i']})$. Then $x = t \cdot x_{[\sigma,i]}$ for some $t \in T$ and $[\sigma, i] \in \Omega(\mathcal{S})$ and hence

$$f(x) = x_{[\sigma',i']} = \varphi(t) \cdot f(x_{[\sigma,i]}).$$

By Lemma 3.4, we know obtain that $f(x_{[\sigma,i]})$ is a distinguished point. This implies

$$f(x_{[\sigma,i]}) = x_{[\sigma',i']} \quad \text{and} \quad \varphi(t) \cdot x_{[\sigma',i']} = x_{[\sigma',i']}. \quad \square$$

Now we come to the main result of this section, namely to generalize the correspondence between fans and toric varieties to a correspondence between systems of fans and toric prevarieties. By construction, $\mathfrak{TP}: \mathcal{S} \mapsto X_{\mathcal{S}}$, $(F, \mathfrak{f}) \mapsto f$ is a covariant functor from the category of systems of fans to the category of toric prevarieties.

3.6 Theorem. *$\mathfrak{TP}$ and the restriction of $\mathfrak{TP}$ to the (full) subcategory of affine systems of fans are equivalences of categories.*

Proof. By equivariance, a toric morphism is determined by its associated homomorphism of the acting tori and its values on the distinguished points. Hence Lemma 3.4 yields that the functor $\mathfrak{TP}$ is faithful.

Next we verify that $\mathfrak{TP}$ is fully faithful. Let $\mathcal{S}$ and $\mathcal{S}'$ be systems of fans in lattices N and N' respectively and let $f: X_{\mathcal{S}} \to X_{\mathcal{S}'}$ be a toric morphism. Then the associated homomorphism $\varphi: T \to T'$ of the respective tori defines a homomorphism $F: N \to N'$.

For $[\sigma, i] \in \Omega(\mathcal{S})$ the associated distinguished point $x_{[\sigma,i]}$ is mapped to a distinguished point $x'_{[\tau,j]}$ and $f(X_{[\sigma,i]}) \subset X'_{[\tau,j]}$ (see Lemma 1.7 and Remark 2.9). Set $\mathfrak{f}([\sigma, i]) := [\tau, j]$. Now it follows from Remark 1.4 and Lemma 2.8 that $(F, \mathfrak{f})$ is a map of systems of fans. By Lemma 3.4, f is the toric morphism associated to $(F, \mathfrak{f})$.

Finally we have to show that for every toric prevariety X there exists an affine system of fans $\mathcal{S}$ with $X \cong X_{\mathcal{S}}$. Let $X_1, \ldots, X_r$ be the maximal T-stable affine charts of X (see Proposition 1.3), and let N denote the lattice of one–parameter–subgroups of the acting torus T of X. Then every chart X_i corresponds to a irreducible fan Δ_{ii} in N.

For every $i, j \in I := \{1, \ldots, r\}$ the intersection $X_i \cap X_j$ is an open toric subvariety of both X_i and X_j and hence corresponds to a fan Δ_{ij} which is a common subfan of Δ_{ii} and Δ_{jj}. It follows that the collection $\mathcal{S} := (\Delta_{ij})_{1 \le i,j \le r}$ forms an affine system of fans and it is straightforward to check that $X \cong X_{\mathcal{S}}$. $\square$

4 The Toric Separation

Let X be a toric prevariety. A *toric separation* of X is a toric morphism $p: X \to Y$ to a toric variety Y that has the following universal property: For every toric morphism f from X to a toric variety Z there exists a unique toric morphism $\widetilde{f}: Y \to Z$ such that $f = \widetilde{f} \circ p$ holds. The main result of this section is

4.1 Theorem. *Every toric prevariety has a toric separation.*

We prove this statement by showing the corresponding result (Theorem 4.2 below) in the category of systems of fans. First we translate the notion of the toric separation into the language of systems of fans: Let N be a lattice and assume that $\mathcal{S}$ is a system of fans in N.

We call a map $(P, \mathfrak{p})$ of systems of fans from $\mathcal{S}$ to a fan $\widetilde{\Delta}$ in a lattice $\widetilde{N}$ a *reduction to a fan*, if for each map $(F, \mathfrak{f})$ of systems of fans from $\mathcal{S}$ to a fan Δ' in a lattice N' there is a unique lattice homomorphism $\widetilde{F}: \widetilde{N} \to N'$ defining a map of the fans $\widetilde{\Delta}$ and Δ' such that $F = \widetilde{F} \circ P$.

4.2 Theorem. *Every system of fans admits a reduction to a fan.*

Proof. Let $\mathcal{S}$ be a system of fans in a lattice N. Then $S := \bigcup_i \Delta_{ii}$ is a system of cones in N in the sense of [1], Section 2. Moreover, every map of systems of fans from $\mathcal{S}$ to a fan Δ' is also a map of the systems S and Δ' of cones.

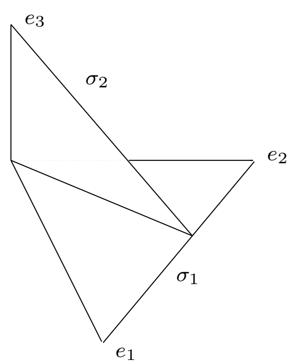
Denote by $\widetilde{\Delta}$ the quotient fan of S by the trivial sublattice $L = \{0\}$ of N (see [1], Definition 2.1 and Theorem 2.3). Then $\widetilde{\Delta}$ lives in some lattice $\widetilde{N}$ and there is a projection $P: N \to \widetilde{N}$, mapping the cones from S into cones of $\widetilde{\Delta}$. By Remark 3.2, P defines a map of the systems of fans $\mathcal{S}$ and $\widetilde{\Delta}$. It follows from the universal property of the quotient fan that P is the reduction of $\mathcal{S}$ to a fan. $\square$

By a *separation* of a prevariety X we mean a regular map p from X to a variety Y that is universal with respect to arbitrary regular maps from X to varieties. It can be shown that every toric prevariety of dimension less than three has such a separation (see [2]).

In dimension three we find the first examples of toric prevarieties that need not have a separation. The remainder of this section is devoted to giving such an example. Let $I = \{1, 2\}$ and let $\mathcal{S}$ be the affine system of fans in $\mathbb{Z}^3$ determined by the cones

$$\sigma_1 := \operatorname{cone}(e_1, e_2), \qquad \sigma_2 := \operatorname{cone}(e_1 + e_2, e_3)$$

glued along 0. Note that $X_{\mathcal{S}}$ is the glueing of $\mathbb{C}^2 \times \mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{C}^2$ along $(\mathbb{C}^*)^3$ via the map $(t_1, t_2, t_3) \mapsto (t_1 t_2^{-1}, t_2, t_3)$.



4.3 Proposition. *$X_{\mathcal{S}}$ admits no separation.*

Proof. Assume that there exists a separation $p: X_{\mathcal{S}} \to Y$. With the universal property we obtain that p is surjective, Y is normal and there is an induced (set theoretical) action of $T := (\mathbb{C}^*)^3$ on Y such that p is equivariant. We lead this to a contradiction by showing that the toric separation of $X_{\mathcal{S}}$ does not factor through p.

First we describe the toric separation of $X_{\mathcal{S}}$ explicitly. Let $\widetilde{\Delta}$ be the fan of faces of $\sigma := \operatorname{cone}(e_1, e_2, e_3)$ in $\mathbb{Z}^3$. The reduction of $\mathcal{S}$ to a fan is the map $Q := \mathrm{id}_{\mathbb{Z}^3}$ of the systems of fans $\mathcal{S}$ and $\widetilde{\Delta}$. Set $Z := X_{\widetilde{\Delta}} = \mathbb{C}^3$. The toric separation of $X_{\mathcal{S}}$ is the toric morphism $q: X_{\mathcal{S}} \to Z$ associated to the map Q of systems of fans. Note that $q(X_{\mathcal{S}})$ is not open in Z, since we have

$$q(X_{\mathcal{S}}) = \mathbb{C}^3 \setminus (\{0\} \times \mathbb{C}^* \times \{0\} \cup \mathbb{C}^* \times \{0\} \times \{0\}).$$

Now, by the universal property of p there is a unique regular map $f: Y \to Z$ such that $q = f \circ p$. Clearly f is T-equivariant. We claim moreover that f is injective. To verify this, we investigate the fibres of f.

Note first that, by equivariance of f, it suffices to consider the fibres of distinguished points. Moreover, by surjectivity of p, we have $f^{-1}(z) = p(q^{-1}(z))$ for every $z \in Z$. Using the Fibre Formula 3.5, we see that

$$q^{-1}(z_0) = x_0, \qquad q^{-1}(z_{\sigma_1}) = x_{\sigma_1},$$

where $\varrho_i := \mathbb{R}_{\ge 0} e_i$. This implies that $f^{-1}(z_0)$ as well as the fibres $f^{-1}(z_{\varrho_i})$ consist of exactly one point. Again by the Fibre Formula one has

$$q^{-1}(z_{\sigma_1}) = \{x_{[\sigma_1,1]}\} \cup H \cdot x_{[\varrho,2]},$$

where $\varrho := \mathbb{R}_{\ge 0}(e_1 + e_2)$ and H is the subtorus of T that corresponds to the sublattice $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ of $\mathbb{Z}^3$. Note that for the one parameter subgroup $\lambda \colon \mathbb{C}^* \to T$ corresponding to the lattice vector $e_1 + e_2$ we obtain

$$\lim_{t \to 0} \lambda(t) \cdot x_0 = x_{[\sigma_1,1]}, \qquad \lim_{t \to 0} \lambda(t) \cdot x_0 = x_{[\varrho,2]}$$

in the affine charts X_{σ_1} and X_{σ_2} respectively. Thus the points $x_{[\sigma_1,1]}$ and $x_{[\varrho,2]}$ cannot be separated by complex open neighbourhoods.

Since p is continuous with respect to the complex topology and Y is Hausdorff it follows $p(x_{[\sigma_1,1]}) = p(x_{[\varrho,2]})$. Since H fixes $x_{[\sigma_1,1]}$ and p is equivariant, H fixes also $p(x_{[\varrho,2]})$. Consequently we obtain that $f^{-1}(z_{\sigma_1})$ consists of a single point.

Finally we have to consider z_σ. We have $q^{-1}(z_\sigma) = T \cdot x_{[\sigma_2,2]}$. In order to see that $f^{-1}(z_\sigma)$ is a single point, it suffices to check that $p(x_{[\sigma_2,2]})$ is fixed by T. Since $q(x_{[\sigma_2,2]}) \ne q(x_{[\varrho,2]})$, we obtain

$$\begin{aligned} T \cdot p(x_{[\sigma_2,2]}) \;&\subset\; \overline{p(T \cdot x_{[\varrho,2]})} \setminus p(T \cdot x_{[\varrho,2]}) \\ &=\; \overline{p(T \cdot x_{[\sigma_1,1]})} \setminus p(T \cdot x_{[\sigma_1,1]}). \end{aligned}$$

Here "$\overline{}$" refers to taking the Zariski closure. Since $\overline{p(T \cdot x_{[\sigma_1,1]})}$ is of dimension one, it follows that $T \cdot p(x_{[\sigma_2,2]})$ is a point, i.e., $p(x_{[\sigma_2,2]})$ is fixed by T.

So we verified that f is injective. Since it is a regular map of normal varieties, Zariski's Main Theorem yields that f is an open embedding. In particular, $f(Y) = q(X)$ is open in Z which is a contradiction. $\square$

5 Some Convex Geometry

Before coming to the investigation of quotients by the action of a subtorus, in this section we recall some elementary properties of convex cones that will be needed later on. Let V be a finite–dimensional real vector space. In Section 7 we will need the following fact:

5.1 Lemma. *Let ϱ and τ be convex cones in V such that $\varrho^\circ \cap \tau \ne \emptyset$. Then τ° is contained in the relative interior of $\sigma := \operatorname{conv}(\tau \cup \varrho)$.*

Proof. Since τ is contained in σ, it suffices to show that $\tau^\circ \cap \sigma^\circ$ is non–empty. Choose $v_1 \in \varrho^\circ \cap \tau$ and $v_2 \in \tau^\circ$. Then $v := v_1 + v_2$ lies in τ°. We claim that $v \in \sigma^\circ$. In order to check this, we have to show that every linear form u contained in the dual cone $\sigma^\vee$ of σ with $u(v) = 0$ vanishes on σ. So let $u \in \sigma^\vee = \varrho^\vee \cap \tau^\vee$ with $u(v) = 0$. Then we obtain that $u(v_1) = u(v_2) = 0$. Consequently u vanishes on ϱ and τ. This yields $u|_\sigma = 0$. $\square$

Let σ denote a convex (polyhedral) cone in V. The set of faces of σ is denoted by $\mathfrak{F}(\sigma)$. Note that the smallest face of σ is $\sigma \cap -\sigma$ and hence equals the maximal linear subspace contained in σ.

We consider the following situation: Let $W \subset V$ be any linear subspace and let $P \colon V \to V/W$ denote the projection. Then $P(\sigma)$ is a cone in V/W. We now want to describe the faces of $P(\sigma)$ in terms of faces of σ. The first statement is the following

5.2 Remark. i) There is an injective map from $\mathfrak{F}(P(\sigma))$ to $\mathfrak{F}(\sigma)$, given by $\widetilde{\tau} \mapsto P^{-1}(\widetilde{\tau}) \cap \sigma$.

ii) If $W \subset (\sigma \cap -\sigma)$, then $\tau \mapsto P(\tau)$ is a bijective map from $\mathfrak{F}(\sigma)$ to $\mathfrak{F}(P(\sigma))$, inverse to the map in i). $\diamondsuit$

Now let σ_W denote the smallest face of σ containing $W \cap \sigma$. Then σ_W is also the largest face of σ with $\sigma_W^\circ \cap W \neq \emptyset$.

5.3 Remark. $\widehat{W} := \sigma_W + W$ is the smallest face of the cone $\sigma + W$. In particular, $\widehat{W}$ is a linear subspace of V. $\diamondsuit$

Let σ, V, W and $\widehat{W}$ be as above. We consider the projections $P^1 \colon V \to V/W$ and $P \colon V \to V/\widehat{W}$. As shown above, $\widehat{W}$ is the smallest face of $\sigma + W$. Hence Remark 5.2 yields $1-1$–correspondences

$$\begin{aligned} \mathfrak{F}(\sigma + W) \to \mathfrak{F}(P^1(\sigma)), &\qquad \tau \mapsto P^1(\tau), \\ \mathfrak{F}(\sigma + W) \to \mathfrak{F}(P(\sigma)), &\qquad \tau \mapsto P(\tau). \end{aligned}$$

In particular, the smallest face of $P^1(\sigma)$ is $\widehat{W}/W$ and $P(\sigma)$ is strictly convex. Moreover, in these notations we have

5.4 Remark. For a given face τ of σ the following conditions are equivalent:

i) $(\tau + W) \prec (\sigma + W)$ and $(\tau + W) \cap \sigma = \tau$.

ii) $P^1(\tau) \prec P^1(\sigma)$ and $P^{1^{-1}}(P^1(\tau)) \cap \sigma = \tau$.

iii) $P(\tau) \prec P(\sigma)$ and $P^{-1}(P(\tau)) \cap \sigma = \tau$. $\diamondsuit$

For later use we introduce here the following generalization of the notion of a fan. Let N denote a lattice and let Σ be a finite set of not necessarily strictly convex cones in N. We call Σ a *quasi–fan*, if $\sigma \in \Sigma$ implies that every face of σ lies in Σ and for any two cones $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face of both, σ and σ'.

For a given quasi–fan Σ in N, let σ_0 denote its minimal element, i.e., σ_0 is the minimal face of each $\sigma \in \Sigma$. Consider the primitive sublattice $L := \sigma_0 \cap N$ of N and let $P \colon N \to N/L$ denote the projection. As an immediate consequence of Remark 5.2 we obtain:

5.5 Remark. The set $\Delta := \{P_{\mathbb{R}}(\sigma);\ \sigma \in \Sigma\}$ is a fan in N/L. $\diamondsuit$

The concept of a system of fans also has a natural generalization in this framework: We call a finite family $\mathcal{S} := (\Sigma_{ij})_{i,j \in I}$ a system of quasi–fans if it satisfies the conditions 2.1 i) to iii). Again such a system of quasi–fans is called affine if each Σ_{ii} is the quasi–fan of faces of a single cone $\sigma(i)$.

As in the case of systems of fans, we define a glueing relation on the set $\mathfrak{F}(\mathcal{S}) := \{(\sigma, i);\ i \in I, \sigma \in \Sigma_{ii}\}$ of labelled faces of a system $\mathcal{S}$ of quasi–fans and denote by $\Omega(\mathcal{S})$ the set of equivalence classes.

A map of two systems $\mathcal{S}$, $\mathcal{S}'$ of quasi–fans in lattices N, N' respectively, is pair $(F, \mathfrak{f})$, where $F \colon N \to N'$ is a lattice homomorphism and $\mathfrak{f} \colon \Omega(\mathcal{S}) \to \Omega(\mathcal{S}')$ is a map that satisfies the conditions of 3.1.

For practical purposes we note that a map of the systems $\mathcal{S}$ and $\mathcal{S}'$ of (quasi–) fans is in certain cases induced by a lattice homomorphism together with a compatible map of the index sets I and I'.

5.6 Lemma. *Let $F\colon N \to N'$ be a lattice homomorphism and let $\mu\colon I \to I'$, $i \mapsto i'$ be a map such that for any two $i, j \in I$ we have*

(∗) *for every $\sigma \in \Sigma_{ij}$ there is a $\sigma' \in \Sigma'_{i'j'}$ with $F_{\mathbb{R}}(\sigma) \subset \sigma'$.*

Then there is a unique map $(F, \mathfrak{f})$ of the systems of quasi–fans $\mathcal{S}$ and $\mathcal{S}'$ with $\mathfrak{f}([\sigma, i]) \prec [\sigma', i']$ for all σ, σ' as in (∗).

Proof. Let $[\sigma, i] \in \Omega(\mathcal{S})$. Choose $\sigma' \in \Sigma'_{i'i'}$ as in Condition (∗) and let σ'' denote the smallest face of σ' with $F_{\mathbb{R}}(\sigma) \subset \sigma''$, in other words σ'' is the face of σ' with $F_{\mathbb{R}}(\sigma^\circ) \subset (\sigma'')^\circ$. Set

$$\mathfrak{f}([\sigma, i]) := [\sigma'', i'] \prec [\sigma', i'].$$

In order to see that $\mathfrak{f}$ is well defined and order preserving, let $[\tau, j] \prec [\sigma, i]$. Choose $\tau'' \in \Sigma'_{j'j'}$ as above. Since $\tau \in \Sigma_{ij}$, Property (∗) yields $\tau'' \in \Sigma'_{i'j'}$ and hence we obtain

$$[\tau'', j'] = [\tau'', i'] \prec [\sigma', i'].$$

To obtain uniqueness of the map $(F, \mathfrak{f})$ of systems of quasi–fans, note that $\mathfrak{f}([\sigma, i]) \prec [\sigma', i']$ readily implies $\mathfrak{f}[\sigma, i] = [\sigma'', i]$, where σ'' is the cone in $\Sigma_{i'i'}$ with $F_{\mathbb{R}}(\sigma^\circ) \subset (\sigma'')^\circ$. □

5.7 Remark. Every map from an affine system of quasi–fans to an arbitrary system of quasi–fans arises from a map of the index sets as in 5.6. ◊

In contrast to this observation maps from general systems of quasi–fans may not have a description by a map of the index sets as the following example shows.

5.8 Example. Let $\mathcal{S} = (\Delta_{11})$ be a single fan consisting of two maximal cones σ_1 and σ_2. Let $I' = \{1, 2\}$ and for $i \in I'$ let Δ'_{ii} denote the fan of faces of σ_i and let $\Delta'_{12} = \Delta'_{21}$ be the fan of faces of $\sigma_1 \cap \sigma_2$. Then the identity of N defines a unique map of systems of fans from $\mathcal{S}$ to $\mathcal{S}' := (\Delta'_{ij})_{i,j \in I'}$. But there is no map from $I = \{1\}$ to I' satisfying (∗). ◊

Now, let $\mathcal{S} = (\Sigma_{ij})_{i,j \in I}$ be a system of quasi–fans in a lattice N. Denote by σ_0 the minimal element of some (and hence all) Σ_{ij}. As above, set $L := \sigma_0 \cap N$ and let $P\colon N \to N/L$ be the projection. Set

$$\Delta_{ij} := \{P_{\mathbb{R}}(\sigma);\ \sigma \in \Sigma_{ij}\}.$$

5.9 Remark. $\widetilde{\mathcal{S}} := (\Delta_{ij})_{i,j \in I}$ is a system of fans in N/L. The system $\widetilde{\mathcal{S}}$ is affine, if and only if $\mathcal{S}$ is affine. Moreover, the map

$$\mathfrak{p}\colon \Omega(\mathcal{S}) \to \Omega(\widetilde{\mathcal{S}}), \qquad [\sigma, i] \mapsto [P_{\mathbb{R}}(\sigma), i]$$

is an order–preserving bijection and $(P, \mathfrak{p})$ is a map of systems of quasi–fans. Any further map from $\mathcal{S}$ to a system of fans factors uniquely through $(P, \mathfrak{p})$. ◊

6 Good Prequotients

Let G be a reductive complex algebraic group. For an algebraic action of G on a variety, Seshadri introduced the notion of a good quotient (see [10], Def. 1.5). His notion can be carried over to the category of prevarieties. Let X be a complex algebraic prevariety and assume that G acts on X by means of a regular map $G \times X \to X$.

6.1 Definition. A G-invariant regular map $p\colon X \to Y$ onto a prevariety Y is called a *good prequotient* for the action of G on X if:

i) p is an affine map, i.e., for every affine open subspace V of Y the open subspace $U := p^{-1}(V)$ of X is affine,

ii) $\mathcal{O}_Y$ is the sheaf $(p_* \mathcal{O}_X)^G$ of invariants, i.e., for every open set $V \subset Y$ we have $\mathcal{O}_Y(V) = \mathcal{O}_X(p^{-1}(V))^G$.

Note that if a good prequotient exists and if both, X and Y, are separated, then the good prequotient is nothing but the good quotient. But in general, even if X is separated, the action of G may admit a good prequotient but no good quotient (see Example 6.10).

As in the case of varieties, a good prequotient is obtained by glueing algebraic quotients of G–stable affine charts.

6.2 Lemma. *A G–invariant surjective regular map $p: X \to Y$ is a good prequotient for the action of H if there is a covering of Y by open affine subspaces V_i, $i \in I$, such that for every $i \in I$ we have*

i) *$U_i := p^{-1}(V_i)$ is an open affine subspace of X,*

ii) *$p|_{U_i}: U_i \to V_i$ is an algebraic quotient for the action of G on U_i, i.e., $p|_{U_i}$ is given by the inclusion $\mathbb{C}[U_i]^G \subset \mathbb{C}[V_i]$.* □

6.3 Definition. A G–invariant regular map $p: X \to Y$ to a complex prevariety Y is called a *categorical prequotient*, if every G–invariant regular map from X to a prevariety factors uniquely through p.

Note that categorical prequotients are necessarily surjective. In analogy to the situation of varieties (see [10], p. 516) one concludes:

6.4 Proposition. *Every good prequotient is a categorical prequotient.* □

Now we specialize to the case that X is a toric prevariety with acting torus T and we consider a subtorus $H \subset T$.

6.5 Corollary. *If $p: X \to Y$ is a good prequotient for the action of H on X then Y is a toric prevariety and p is a toric morphism.*

Proof. Choose a covering of Y by open affine subspaces V_i such that the conditions i) and ii) of Lemma 6.2 are satisfied. Consider the action of H on $T \times X$ defined by $h{\cdot}(t, x) := (t, h{\cdot}x)$. The map

$$q := \mathrm{id}_T \times p: T \times X \to T \times Y$$

is a good prequotient for this action, since the sets $T \times V_i$ satisfy the conditions of Lemma 6.2. By Proposition 6.4, we obtain a commutative diagram of regular maps

$$\begin{array}{ccc} T \times X & \longrightarrow & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ T \times Y & \longrightarrow & Y \end{array} ,$$

where the horizontal arrows indicate regular T-actions. Since Y is a normal prevariety the claim follows. □

In Theorem 3.6 we showed that every toric prevariety arises from an affine system of fans. As the main result of this section we characterize in terms of affine systems of fans,

when the action of a subtorus on the toric prevariety admits a good prequotient. For the corresponding statements on toric varieties we refer to [12] and [8].

Let us first recall the description of the good quotient in the affine case. Consider an affine toric variety X_σ where σ is a strictly convex cone in the lattice N and let L be the primitive sublattice of N corresponding to a subtorus H of the acting torus of X_σ.

Let $\sigma_L := \sigma_{L_{\mathbb{R}}}$, i.e., σ_L is the largest face of σ with $L_{\mathbb{R}} \cap \sigma_L^\circ \neq \emptyset$ (see also Section 5). Let $\widehat{L} := N \cap (L_{\mathbb{R}} + \sigma_L)$. Denote by $P\colon N \to N/\widehat{L}$ the projection. Then $\widetilde{\sigma} := P_{\mathbb{R}}(\sigma)$ is a strictly convex cone in $N/\widehat{L}$. Moreover, we have (see e.g. [1], Example 3.1):

6.6 Remark. The toric morphism $X_\sigma \to X_{\widetilde{\sigma}}$ associated to P is the algebraic quotient for the action of H on X_σ. $\diamondsuit$

Now we formulate our criterion for the general case. Let N be a lattice, let I be a finite index set, and let $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ be an affine system of fans in N. Recall that for every i there is a strictly convex cone $\sigma(i)$ such that Δ_{ii} is the fan of faces of $\sigma(i)$.

Let $X_{\mathcal{S}}$ be the toric prevariety associated to $\mathcal{S}$ and let H be a subtorus of its acting torus. Let L denote the (primitive) sublattice of N corresponding to H and let $P^1\colon N \to N/L$ denote the projection. With these notations our result is the following:

6.7 Theorem. *The action of H on $X_{\mathcal{S}}$ admits a good prequotient if and only if for every $i, j \in I$ and every $\tau \in \Delta_{ij}^{\max}$ the following holds:*

i) $P^1_{\mathbb{R}}(\tau) \prec P^1_{\mathbb{R}}(\sigma(i))$,

ii) ${P^1_{\mathbb{R}}}^{-1}(P^1_{\mathbb{R}}(\tau)) \cap \sigma(i) = \tau$.

If these conditions are satisfied then there is a primitive sublattice $\widehat{L}$ of L such that $\widehat{L}_{\mathbb{R}} = L_{\mathbb{R}} + \sigma(i)_L$ for all i.

In the proof of this result we use the following description of affine toric morphisms. Let $(F, \mathfrak{f})$ be a map of systems of fans from $\mathcal{S}$ to an affine system of fans $\mathcal{S}' := (\Delta'_{i'j'})_{i',j' \in I'}$ in some lattice N'.

6.8 Lemma. *The toric morphism f is affine if and only if for every $i' \in I'$ the set*

$$R(i') := \{[\tau, j] \in \Omega(\mathcal{S});\ \mathfrak{f}([\tau, j]) \prec [\sigma'(i'), i']\}$$

contains a unique maximal element.

Proof. Let $i' \in I'$. Using Lemma 2.8 and the Fibre Formula 3.5, we obtain the following formula for the preimage of the maximal affine chart $X'_{i'} := X_{[\sigma'(i'), i']}$ of $X_{\mathcal{S}'}$:

$$f^{-1}(X'_{i'}) = \bigcup_{\mathfrak{f}([\tau,j]) \in R(i')} T \cdot x_{[\tau,j]}\,.$$

This open subspace of X is an affine variety if and only if $R(i')$ contains a unique maximal element. This proves the claim. $\square$

Proof of Theorem 6.7. Assume first that the conditions i) and ii) are valid. For $i \in I$ set $\widetilde{\sigma}(i) := P^1_{\mathbb{R}}(\sigma(i))$. Note that $\widetilde{\sigma}(i)$ equals $P^1_{\mathbb{R}}(\sigma(i) + L_{\mathbb{R}})$. Thus, by Remark 5.2, the smallest face of $\widetilde{\sigma}(i)$ is

$$P^1_{\mathbb{R}}(\sigma(i)_L + L_{\mathbb{R}}) = P^1_{\mathbb{R}}(\sigma(i)_L).$$

Let Σ_{ii} denote the quasi–fan of faces of $\widetilde{\sigma}(i)$. For $i, j \in I$ define Σ_{ij} to be the set of all faces of the cones $P^1_{\mathbb{R}}(\tau)$, $\tau \in \Delta_{ij}^{\max}$. Then it follows from Condition i) that Σ_{ij} is in fact a sub–quasi–fan of Σ_{ii}.

We claim that $(\Sigma_{ij})_{i,j\in I}$ is a system of quasi–fans. To show this we need to verify $\Sigma_{ij} \cap \Sigma_{jk} \subset \Sigma_{ik}$. Suppose that $\widetilde{\varrho} \in \Sigma_{ij} \cap \Sigma_{jk}$, i.e., $\widetilde{\varrho}$ is a face of a cone $\widetilde{\tau} := P^1_{\mathbb{R}}(\tau) \cap P^1_{\mathbb{R}}(\tau')$ with some $\tau \in \Delta^{\max}_{ij}$ and $\tau' \in \Delta^{\max}_{jk}$. By Condition ii), we have

$$(P^1_{\mathbb{R}})^{-1}(\widetilde{\tau}) \cap \sigma(j) = \tau \cap \tau',$$

and in particular, $\widetilde{\tau} = P^1_{\mathbb{R}}(\tau \cap \tau')$. Since $\tau \cap \tau'$ lies in $\Delta_{ij} \cap \Delta_{jk} \subset \Delta_{ik}$ and since $\widetilde{\tau}$ is a common face of $\widetilde{\sigma}(i)$ and $\widetilde{\sigma}(k)$, we can conclude that $\widetilde{\tau} \in \Sigma_{ik}$ and hence $\widetilde{\varrho} \in \Sigma_{ik}$.

By construction, the maps P^1 and $\mu := \mathrm{id}_I$ satisfy the assumptions of Lemma 5.6. Hence they determine a unique map of systems of quasi–fans $(P^1, \mathfrak{p}^1)$ from $\mathcal{S}$ to $(\Sigma_{ij})_{i,j\in I}$ with $\mathfrak{p}^1([\sigma(i), i]) = [\widetilde{\sigma}(i), i]$.

Since in a system of quasi–fans the minimal elements of the Σ_{ij} all coincide, Remark 5.2 yields $\sigma(i)_L + L_{\mathbb{R}} = \sigma(j)_L + L_{\mathbb{R}}$ for all i, j. So there is a primitive sublattice $\widehat{L}$ of L such that $\widehat{L}_{\mathbb{R}} = L_{\mathbb{R}} + \sigma(i)_L$ for all $i \in I$.

Let $Q\colon N/L \to \widetilde{N} := N/\widehat{L}$ denote the projection. According to Remark 5.9 the sets $\widetilde{\Delta}_{ij} := \{Q_{\mathbb{R}}(\tau);\ \tau \in \Sigma_{ij}\}$ form a system $\widetilde{\mathcal{S}}$ of fans in $\widetilde{N}$ and the map $(Q, \mathfrak{q})$ with $\mathfrak{q}\colon [\tau, i] \mapsto [Q_{\mathbb{R}}(\tau), i]$ is universal with respect to maps to systems of fans.

Let $p\colon X_{\mathcal{S}} \to X_{\widetilde{\mathcal{S}}}$ denote the toric morphism associated to $(Q \circ P^1, \mathfrak{q} \circ \mathfrak{p})$. Since the conditions of Lemma 6.8 are satisfied, the morphism p is affine. Moreover, by Remark 6.6, for every $i \in I$ the restriction $p|_{X_i}\colon X_i \to \widetilde{X}_i$ is the algebraic quotient for the action of H. Now it follows from Lemma 6.2 that p is a good prequotient for the action of H on X.

Conversely, let $p\colon X \to \widetilde{X}$ be a good prequotient for the action of H on X. By Corollary 6.5 and Theorem 3.6, we may assume that p arises from a map $(P, \mathfrak{p})$ of affine systems of fans $\mathcal{S}$ in N and $\widetilde{\mathcal{S}}$ in $\widetilde{N}$.

The restriction $p_1\colon p^{-1}(\widetilde{T}) \to \widetilde{T}$ of p is an algebraic quotient of affine toric varieties. So, since P is the lattice homomorphism associated to p_1, Remark 6.6 implies that P is surjective. Therefore, setting $\widehat{L} := \ker(P)$, we can assume that $\widetilde{N} = N/\widehat{L}$ and P is the canonical projection.

Since p is an affine surjective toric morphism, we can assume by Lemma 6.8 and the fibre formula that $I = \widetilde{I}$ and $\mathfrak{p}([\sigma(i), i]) = [\widetilde{\sigma}(i), i]$ with $\widetilde{\sigma}(i) = P_{\mathbb{R}}(\sigma(i))$ hold. Moreover, since the $X_{[\sigma(i),i]}$ are maximal T–stable affine open subspaces of X, we have

$$X_{[\sigma(i),i]} = p^{-1}(\widetilde{X}_{[\widetilde{\sigma}(i),i]}).$$

Since the restriction of p to $X_{[\sigma(i),i]}$ is an algebraic quotient for the action of H, Remark 6.6 yields $\sigma(i)_L + L_{\mathbb{R}} = \sigma(j)_L + L_{\mathbb{R}}$ any two $i, j \in I$.

Now consider $\tau \in \Delta^{\max}_{ij}$ for some $i, j \in I$. Since $(P, \mathfrak{p})$ is a map of systems of fans, there is a cone $\widetilde{\tau} \in \widetilde{\Delta}_{ij}$ such that $\mathfrak{p}([\tau, i]) = [\widetilde{\tau}, i]$. On the other hand, $\widetilde{\tau} \prec P_{\mathbb{R}}(\sigma(i))$ implies by Remark 5.2 that $\sigma := P^{-1}_{\mathbb{R}}(\widetilde{\tau}) \cap \sigma(i)$ is a face of $\sigma(i)$. Clearly we have $\tau \prec \sigma$.

Since we have $\mathfrak{p}([\sigma, i]) = [\widetilde{\tau}, i] \prec [\widetilde{\sigma}(j), j]$, Lemma 6.8 yields $[\sigma, i] \prec [\sigma(j), j]$ and hence $\sigma \in \Delta_{ij}$. That implies $\tau = \sigma$ and we obtain that $P_{\mathbb{R}}(\tau) = \widetilde{\tau}$ and $P^{-1}_{\mathbb{R}}(P_{\mathbb{R}}(\tau)) \cap \sigma(i) = \tau$. As a consequence of Remark 5.4 we get conditions i) and ii). $\square$

6.9 Corollary. *Let $p\colon X_{\mathcal{S}} \to X_{\mathcal{S}'}$ be a surjective affine toric morphism of prevarieties. If the homomorphism of the acting tori associated to p has a connected kernel H, then p is a good prequotient for the action of H on $X_{\mathcal{S}}$.*

Proof. We may assume that p arises from a map $(P, \mathfrak{p})$ of systems of fans. Since H was assumed to be connected, P is surjective and hence a projection. By Lemma 6.8 we can assume that $P_{\mathbb{R}}(\sigma(i)) = \sigma'(i)$ and $\mathfrak{p}([\sigma(i), i]) = [\sigma'(i), i]$.

Let $\tau \in \Delta_{ij}^{\max}$ for some $i \neq j$. Then there is a cone $\tau' \in \Delta'_{ij}$ such that $\mathfrak{p}([\tau, i]) = [\tau', j]$. Since $\tau' \prec \sigma'(i)$, we have $\sigma := P_{\mathbb{R}}^{-1}(\tau') \cap \sigma(i) \prec \sigma(i)$ and $\mathfrak{p}([\sigma, i]) = [\tau', j] \prec [\sigma'(j), j]$. By Lemma 6.8, $\sigma \in \Delta_{ij}$ and hence $\sigma = \tau$. That proves the claim. $\square$

If a toric variety X admits a good quotient $p: X \to Y$ for the action of some subtorus H then by definition, p is also a good prequotient for the action of H. The converse of this statement does not hold, as we see in the following simple example.

6.10 Example. The toric variety $X := \mathbb{C}^2 \backslash \{0\}$ is described by the affine system $\mathcal{S} = (\Delta_{ij})$ of fans in $\mathbb{Z}^2$, where Δ_{ii} for $i = 1, 2$ denotes the fan of faces of $\sigma(1) := \mathbb{R}_{\geq 0} e_1$ and $\sigma(2) := \mathbb{R}_{\geq 0} e_2$ and $\Delta_{12} = \{\{0\}\}$. Consider the subtorus $H := \{(t, t^{-1});\ t \in \mathbb{C}^*\}$ of the acting torus $(\mathbb{C}^*)^2$ of X. Then H corresponds to the sublattice L in $\mathbb{Z}^2$ generated by $e_1 - e_2$.

The projection $P: \mathbb{Z}^2 \to \mathbb{Z}$, $(x, y) \mapsto x + y$ defines an H-invariant toric morphism p from X onto the complex line with doubled zero with $p(z, w) = zw$ for $z, w \neq 0$. The morphism p is a good prequotient for the action of H, but there is no good quotient for the action of H (see e.g. [12] or [8]). $\diamondsuit$

We conclude this section with two further examples, showing that both conditions of Theorem 6.7 are actually needed.

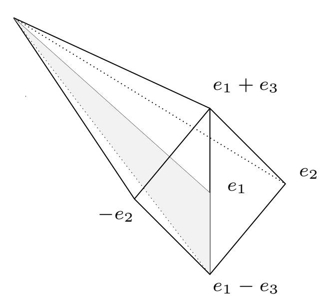
6.11 Example. Consider as in Example 6.10 the affine system of fans $\mathcal{S}$ in $\mathbb{Z}^2$ defining the toric variety $X := \mathbb{C}^2 \setminus \{0\}$. Let $L := \mathbb{R} e_1$. Then the subtorus H corresponding to L equals $\mathbb{C}^* \times \{1\}$. The associated projection of lattices is $P^1: \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto y$. Property 6.7 i) is valid but 6.7 ii) is not. And indeed, the toric morphism $\mathbb{C}^2 \backslash \{0\} \to \mathbb{C}$, $(z, w) \mapsto w$ associated to P^1 is a toric prequotient but not a good quotient. $\diamondsuit$

6.12 Example. Let $\mathcal{S}$ be the affine system of fans in $\mathbb{Z}^6$ obtained from

$$\sigma(1) := \operatorname{cone}(e_1, \ldots, e_4), \qquad \sigma_2 := \operatorname{cone}(e_1, e_4, e_5, e_6)$$

by defining $\Delta_{ii} := \mathfrak{F}(\sigma(i))$ and $\Delta_{12} := \mathfrak{F}(\sigma(1) \cap \sigma(2))$. Define a projection $P^1: \mathbb{Z}^6 \to \mathbb{Z}^3$ by

$$\begin{array}{lll} P^1(e_1) := e_1, & P^1(e_2) := e_1 + e_3, & P^1(e_3) := -e_2, \\ P^1(e_4) := e_1 - e_3, & P^1(e_5) := e_2, & P^1(e_6) := e_1 + e_3. \end{array}$$



Let $L := \ker(P^1)$. Note that

$$P^1_{\mathbb{R}}(\sigma(1)) = \operatorname{cone}(e_1 + e_3, -e_2, e_1 - e_3), \qquad P^1_{\mathbb{R}}(\sigma(2)) = \operatorname{cone}(e_1 + e_3, e_2, e_1 - e_3).$$

Thus we see that for the face $\tau := \operatorname{cone}(e_1, e_4) \in \Delta_{12}^{\max}$ Property 6.7 i) is not valid. However, Property 6.7 ii) holds. $\diamondsuit$

7 The Toric Prequotient

For actions of subtori of the acting torus of a toric variety, we introduced in [1] the notion of a toric quotient. The analogous concept for the action of a subtorus H of the acting torus of a toric prevariety X is the following:

7.1 Definition. An H–invariant toric morphism $p: X \to Y$ to a toric prevariety Y is called a *toric prequotient* for the action of H on X if for every H-invariant toric morphism f from X to a toric prevariety Z there is a unique toric morphism $\widetilde{f}: Y \to Z$ such that $f = \widetilde{f} \circ p$.

If $p: X \to Y$ is a toric prequotient for the action of a subtorus H of the acting torus of X, then the toric prevariety Y is unique up to isomorphy and will also be denoted by $X \mathbin{/_{\mathrm{tpq}}} H$. As a consequence of Proposition 6.4 and Corollary 6.5, every good prequotient is a toric prequotient. The aim of this section is to give a constructive proof for the following

7.2 Theorem. *Every subtorus action on a toric prevariety admits a toric prequotient.*

In view of Theorem 3.6, we prove this result in terms of affine systems of fans. For the translation of the universal property of the toric prequotient into the language of systems of fans we observe:

7.3 Remark. Let $(F, \mathfrak{f})$ be a map of systems of fans $\mathcal{S}$, $\mathcal{S}'$ in lattices N, N' respectively, and let H be a subtorus of the acting torus T of $X_{\mathcal{S}}$. Then the toric morphism $f: X_{\mathcal{S}} \to X_{\mathcal{S}'}$ determined by $(F, \mathfrak{f})$ is H–invariant if and only if the sublattice $L \subset N$ corresponding to H is contained in $\ker(F)$. $\diamondsuit$

Now let N be a lattice and let $\mathcal{S}$ be an affine system of quasi–fans in N. Moreover, let L be a primitive sublattice of N. Then the analogue of Definition 7.1 is the following:

7.4 Definition. A *prequotient* for $\mathcal{S}$ by L is a map of systems of quasi–fans $(P, \mathfrak{p})$ from $\mathcal{S}$ to an affine system $\widetilde{\mathcal{S}}$ of quasi–fans in a lattice $\widetilde{N}$ such that:

i) $L \subset \ker(P)$.

ii) For every map $(F, \mathfrak{f})$ from $\mathcal{S}$ to an affine system of quasi–fans $\mathcal{S}'$ with $F|_L = 0$, there is a unique map $(\widetilde{F}, \widetilde{\mathfrak{f}})$ of the systems of quasi–fans $\widetilde{\mathcal{S}}$ and $\mathcal{S}'$ such that $(F, \mathfrak{f}) = (\widetilde{F}, \widetilde{\mathfrak{f}}) \circ (P, \mathfrak{p})$.

By Remark 5.9, for every affine system $\widetilde{S}$ of quasi–fans in a lattice $\widetilde{N}$ we have a map that is universal with respect to maps from $\widetilde{S}$ to affine systems of fans. Thus Theorem 7.2 follows directly from Theorem 3.6 and the following

7.5 Theorem. *There is an algorithm to construct for a given affine system of quasi–fans $\mathcal{S}$ in N and a primitive sublattice L of N the prequotient of $\mathcal{S}$ by L.*

For the proof of this theorem we introduce the following notion. Let I be a finite index set. We call a collection $\mathfrak{S} := (S_{ij})_{i,j \in I}$ of finite sets of cones in N a *system of related cones* in N if the following conditions are satisfied:

i) S_{ii} contains precisely one maximal cone $\sigma(i)$,

ii) $S_{ij} = S_{ji}$ for all $i, j \in I$,

iii) $\tau \in S_{ij}$ implies $\tau \subset \sigma(i) \cap \sigma(j)$,

iv) If $\tau \in S_{ij}$ then S_{ij} also contains all the faces of τ.

A *map of two systems of related cones* $\mathfrak{S}$ and $\mathfrak{S}'$ in lattices N and N' respectively is a pair (F, μ), where $F\colon N \to N'$ is a lattice homomorphism and $\mu\colon I \to I'$, $i \mapsto i'$ is a map of the index sets of $\mathfrak{S}$ and $\mathfrak{S}'$ such that

($*$) for every $\tau \in S_{ij}$ there is a $\tau' \in S'_{i'j'}$ with $F_{\mathbb{R}}(\tau) \subset \tau'$.

Note that every affine system $\mathcal{S}$ of quasi–fans in N is a system of related cones in N. For two affine systems of quasi–fans $\mathcal{S}$ and $\mathcal{S}'$ in N and N' respectively, any map (F, μ) from $\mathcal{S}$ to $\mathcal{S}'$ as map of systems of related cones uniquely determines a map $(F, \mathfrak{f})$ from $\mathcal{S}$ to $\mathcal{S}'$ as map of systems of fans such that

$$\mathfrak{f}([\sigma, i]) \prec [\sigma'(i'), i']$$

holds for all $[\sigma, i] \in \Omega(\mathcal{S})$ (see Lemma 5.6) and every map of affine systems of quasi–fans arises in this way. But a given $(F, \mathfrak{f})$ can arise from different maps of the systems $\mathcal{S}$ and $\mathcal{S}'$ of related cones.

Proof of Theorem 7.5. Let $\mathcal{S} = (\Sigma_{ij})_{i,j \in I}$ be an affine system of quasi–fans in N and let L be a primitive sublattice of N. We use the following procedure for the calculation of the prequotient of $\mathcal{S}$ by L:

Initialization: Set $\widetilde{N} := N/L$ and let $P\colon N \to \widetilde{N}$ denote the projection. For every $i \in I$ set $\tau^1(i) := P_{\mathbb{R}}(\sigma(i))$. For $i, j \in I$ let S^1_{ij} denote the set of faces of the cones $P_{\mathbb{R}}(\varrho)$, $\varrho \in \Sigma_{ij}^{\max}$. Set $\mathfrak{S}^1 := (S^1_{ij})_{i,j \in I}$.

Loop 1: While there are $i, j \in I$, $\varrho \in {S^1_{ij}}^{\max}$, where "max" refers to the face relation, with $\varrho \not\prec \tau^1(i)$ do the following: Let ϱ_i denote the face of $\tau^1(i)$ with $\varrho^\circ \subset \varrho_i^\circ$. Replace $\tau^1(j)$ by $\mathrm{conv}(\tau^1(j) \cup \varrho_i)$ and replace S^1_{jj} by the set of faces of $\mathrm{conv}(\tau^1(j) \cup \varrho_i)$. Remove $\{\varrho';\ \varrho' \prec \varrho\}$ from S^1_{ij}, S^1_{ji} and add instead $\{\varrho';\ \varrho' \prec \varrho_i\}$.

Loop 2: While there are $i, j, k \in I$ and $\varrho \in S^1_{ij} \cap S^1_{jk}$ such that $\varrho \notin S^1_{ik}$, replace S^1_{ki} and S^1_{ik} by $S^1_{ik} \cup \{\varrho'; \varrho' \prec \varrho\}$.

Output: For every $i, j \in I$ let $\widetilde{\tau}(i) := \tau^1(i)$ and $\widetilde{\Sigma}_{ij} := S^1_{ij}$. Set $\widetilde{\mathcal{S}} := (\widetilde{\Sigma}_{ij})_{i,j \in I}$.

In order to check that the output is in fact well–defined, we have to show that the loops of the algorithm are finite. This is clear for Loop 2. For Loop 1 we use a similar argument as in [1], proof of Theorem 2.3:

Since for each $i, j \in I$ the number of maximal cones of S^1_{ij} does not increase when carrying out a step of Loop 1, it stays fixed after finitely many, say K, steps of Loop 1. Let $E \subset N$ be a minimal set of generators for the cones $\sigma(i)$, $i \in I$. Then in each step after the first K steps the number

$$\sum_{i,j \in I} \sum_{\tau \in {S^1_{ij}}^{\max}} |P(E) \cap \tau|$$

is properly enlarged. This can happen only a finite number of times, i.e., Loop 1 is finite. Thus we obtain that the outputs are in fact well–defined.

Claim: $\widetilde{\mathcal{S}}$ is an affine system of quasi–fans in $\widetilde{N}$. Moreover, (P, id_I) is a map of systems of related cones from $\mathcal{S}$ to $\widetilde{\mathcal{S}}$ and hence defines a map $(P, \mathfrak{p})$ of the systems of quasi–fans $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ such that $\mathfrak{p}([\sigma, i]) \prec [\widetilde{\tau}(i), i]$ for all $[\sigma, i] \in \Omega(\mathcal{S})$. The map $(P, \mathfrak{p})$ is the prequotient for $\mathcal{S}$ by L.

We prove this claim: After leaving Loop 1, every S^1_{ii} is the quasi–fan of faces of $\tau^1(i)$ and every S^1_{ij} is a sub–quasi–fan of the quasi–fan of common faces of $\tau^1(i)$ and $\tau^1(j)$: $S^1_{ij} \prec S^1_{ii} \cap S^1_{jj}$. Note that this property is not affected in Loop 2.

Thus the quasi–fans $\widetilde{\Sigma}_{ij}$ satisfy Properties i) and ii) Definition 2.1. The transitivity axiom iii) is guaranteed by Loop 2. In other words, $\widetilde{\mathcal{S}}$ is an affine system of quasi–fans.

By construction, (P, id_I) is a map of the systems $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ of related cones. Hence there is a unique map $(P, \mathfrak{p})$ of the systems of quasi–fans $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ with

$$\mathfrak{p}([\sigma, i]) \prec [\widetilde{\tau}(i), i]$$

for all $[\sigma, i] \in \Omega(\mathcal{S})$. We have to prove that $(P, \mathfrak{p})$ satisfies the universal property of the prequotient of $\mathcal{S}$ by L. So, let $(F, \mathfrak{f})$ be a map from $\mathcal{S}$ to an affine system $\mathcal{S}' = (\Sigma'_{ij})_{i,j \in I'}$ of quasi–fans in a lattice N' such that $L \subset \ker(F)$. Then there is a lattice homomorphism $\widetilde{F}\colon \widetilde{N} \to N'$ with $F = \widetilde{F} \circ P$.

Now choose a map $\mu\colon I \to I'$, $i \mapsto i'$ such that $\mathfrak{f}([\sigma(i), i]) \prec [\sigma'(i'), i']$. Then $(\widetilde{F}, \mu)$ is a map from the system of related cones $\mathfrak{S}^1$, defined as in the initialization, to $\mathcal{S}'$ such that

$$(F, \mu) = (\widetilde{F}, \mu) \circ (P, \mathrm{id}_I)\,.$$

We show inductively that $(\widetilde{F}, \mu)$ remains a map of lists of related cones, when $\mathfrak{S}^1$ is modified in one of the two loops.

Suppose we are in Loop 1 and there are $i, j \in I$ and $\varrho \in S^{1\,\max}_{ij}$ with $\varrho \not\prec \tau^1(i)$. Let ϱ_i denote the smallest face of $\tau^1(i)$ containing ϱ. Then $\varrho^\circ \subset \varrho_i^\circ$. By the induction hypothesis, there is a cone $\varrho' \in \Sigma'_{i'j'}$ with $\widetilde{F}_{\mathbb{R}}(\varrho) \subset \varrho'$.

We claim that we even have $\widetilde{F}_{\mathbb{R}}(\varrho_i) \subset \varrho'$. To see this note that $\widetilde{F}_{\mathbb{R}}(\varrho_i) \subset \widetilde{F}_{\mathbb{R}}(\tau^1(i)) \subset \sigma'(i')$. So there is a face σ' of $\sigma'(i')$ such that $\widetilde{F}_{\mathbb{R}}(\varrho_i^\circ) \subset \sigma'^\circ$. On the other hand, $\widetilde{F}_{\mathbb{R}}(\varrho^\circ) \subset \varrho' \cap \sigma'^\circ \neq \emptyset$. That implies $\sigma' \prec \rho'$ and the claim follows.

Consequently, we obtain $\widetilde{F}_{\mathbb{R}}(\tau^1(j) \cup \varrho_i) \subset \sigma'(j')$. So the compatibility condition $(*)$ for $(\widetilde{F}, \mu)$ remains true after replacing $\tau^1(j)$ by $\mathrm{conv}(\tau^1(j) \cup \varrho_j)$, S^1_{jj} by the set of faces of $\mathrm{conv}(\tau^1(j) \cup \varrho_j)$ and, in S^1_{ij}, S^1_{ji}, the faces of ϱ by those of ϱ_i.

Now consider Loop 2 and suppose that there are $i, j, k \in I$ and $\varrho \in S^1_{ij} \cap S^1_{jk}$ such that $\varrho \notin S^1_{ik}$. By induction hypothesis, there are cones $\varrho' \in \Sigma'_{i'j'}$ and $\varrho'' \in \Sigma'_{j'k'}$ such that $\widetilde{F}_{\mathbb{R}}(\varrho) \subset \varrho' \cap \varrho''$.

Since ϱ' and ϱ'' are faces of $\sigma'(j')$ they intersect in a common face and in particular, $\varrho' \cap \varrho''$ lies in $\Sigma'_{i'j'} \cap \Sigma'_{j'k'}$ and hence in $\Sigma'_{i'k'}$. This shows that $(*)$ for $(\widetilde{F}, \mu)$ remains true after adding ϱ and all its faces to S^1_{ik} and S^1_{ki}.

In other words the pair $(\widetilde{F}, \mu)$ is a map of the systems $\widetilde{\mathcal{S}}$ and $\mathcal{S}'$ of related cones. Moreover, by definition we have

$$(F, \mu) = (\widetilde{F}, \mu) \circ (P, \mathrm{id}_I)\,.$$

Consequently the associated map of prefans $(\widetilde{F}, \widetilde{\mathfrak{f}})$ from $\widetilde{\mathcal{S}}$ to $\mathcal{S}'$ is a factorization of $(F, \mathfrak{f})$ through $(P, \mathfrak{p})$. We have to check that $(\widetilde{F}, \widetilde{\mathfrak{f}})$ is uniquely determined by this property.

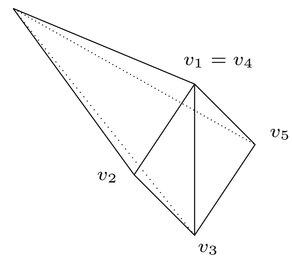
Since P is surjective, $\widetilde{F}$ is determined by $F = \widetilde{F} \circ P$. Note that, before entering Loop 1 for each $i \in I$, the image $P_{\mathbb{R}}(\sigma(i)^\circ)$ is contained in $\tau^1(i)^\circ$. By Lemma 5.1 this property remains valid after enlarging the cones $\tau^1(i)$ as in Loop 1, i.e., we have in fact

$$\mathfrak{p}([\sigma(i), i]) = [\widetilde{\tau}(i), i]$$

holds for every $i \in I$. Consequently one obtains $\widetilde{\mathfrak{f}}([\widetilde{\tau}(i), i]) = \mathfrak{f}([\sigma(i), i])$ for each $i \in I$. Since $\widetilde{\mathfrak{f}}$ is order–preserving, it is already determined by this property. $\square$

7.6 Remark. The toric prequotient for the action of H on $X_{\mathcal{S}}$ is good if and only if the algorithm for constructing the prequotient of $\mathcal{S}$ by the sublattice L corresponding to H already terminates after the initialization and Property 6.7 ii) holds. $\diamondsuit$

7.7 Example. Let $\mathcal{S}$ be the affine system of fans in $\mathbb{Z}^5$ with $\sigma(1) := \mathrm{cone}(e_1, \ldots, e_4)$ and $\sigma(2) := \mathrm{cone}(e_3, e_4, e_5)$ and the maximal glueing relation. Define a lattice homomorphism $P\colon \mathbb{Z}^5 \to \mathbb{Z}^3$ by $P(e_1) := v_i$ where the vectors v_i are situated as indicated below.



Clearly we may arrange the v_i in such a manner that P is surjective. Then for $\tau := \mathrm{cone}(e_3, e_4) \in \Delta_{12}^{\max}$ we obtain

$$P_{\mathbb{R}}^{-1}(P_{\mathbb{R}}(\tau)) \cap \sigma(1) = \mathrm{cone}(e_1, e_3, e_4)$$

and consequently Property 6.7 ii) is not valid. However, in this situation, the algorithm terminates after the initialization. $\diamondsuit$

As Example 6.10 indicates, the toric prequotient of a subtorus action on a toric variety in general differs from its toric quotient. The two notions are related to each other by the toric separation (see Section 4):

7.8 Remark. For the action of a subtorus H of the acting torus of a toric variety X, let $p\colon X \to X \underset{\mathrm{tpq}}{/} H$ be the toric prequotient and let $q\colon X \underset{\mathrm{tpq}}{/} H \to Y$ be the toric separation. Then $q \circ p$ is the toric quotient for the action of H on X. $\diamondsuit$

In particular, the toric prequotient occurs as an intermediate step in the construction of the toric quotient. We conclude this section with an explicit example, showing that both loops of the algorithm are actually needed.

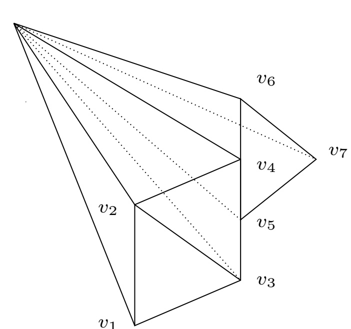
7.9 Example. Let us consider the following three cones in $N = \mathbb{Z}^7$:

$$\sigma(1) := \mathrm{cone}(e_1, e_2, e_3), \quad \sigma(2) := \mathrm{cone}(e_2, e_3, e_4, e_5), \quad \sigma(3) := \mathrm{cone}(e_4, e_5, e_6, e_7).$$

Let $\mathcal{S}$ denote the system of fans with these maximal cones such that:

$$\Delta_{12}^{\max} = \sigma(1) \cap \sigma(2), \quad \Delta_{23}^{\max} = \sigma(2) \cap \sigma(3), \quad \Delta_{13} = \{0\}.$$

Let $P\colon \mathbb{Z}^7 \to \mathbb{Z}^3$ denote the homomorphism given by $P(e_i) := v_i$, where the v_i are vectors in $\mathbb{Z}^3$ situated as in the picture below.



As before we may assume that P is surjective. Then after running through Loop 1 of the prequotient algorithm for $\mathcal{S}$ by L we have the following cones in $\mathfrak{S}^1$:

$$\tau^1(1) = P_{\mathbb{R}}(\sigma(1)), \quad \tau^1(2) = \operatorname{cone}(v_2, v_3, v_6), \quad \tau^1(3) = \operatorname{cone}(v_3, v_6, v_7)\,.$$

and for the families S^1_{ij} of subcones we obtain

$$S^{1\,\max}_{12} = \operatorname{cone}(v_2, v_3), \quad S^{1\,\max}_{23} = \operatorname{cone}(v_3, v_6), \quad S^{1\,\max}_{13} = \{0\}\,.$$

So we see that $\operatorname{cone}(e_3)$ is contained in $S_{12} \cap S_{23}$ but not in S_{13}. Consequently the algorithm also enters Loop 2, where S_{13} is replaced by $\{\operatorname{cone}(e_3), \{0\}\}$. After this the algorithm terminates. $\diamondsuit$

8 Toric Prevarieties as Prequotients of Quasi-Affine Toric Varieties

In [5] it is shown that every toric variety occurs as the image of a good quotient of an open subset of some $\mathbb{C}^s$ by a reductive abelian group H. In fact, a slight modification of Cox's construction yields that any given toric variety is even the image of a good toric quotient of an open subset of some $\mathbb{C}^s$ by a subtorus of $(\mathbb{C}^*)^s$ (see e.g. [4], Section 1).

In this section we make related statements in the setting of toric prevarieties. Let $X_{\mathcal{S}}$ be a toric prevariety arising from a system of fans $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ in a lattice N. We assume $\mathcal{S}$ to be affine. According to Theorem 3.6, this means no loss of generality. Moreover, we may assume that $\mathcal{S}$ is irredundant in the following sense: If $i \neq j$, then Δ_{ij} is a proper subfan of Δ_{ii}.

A first aim is to show that $X_{\mathcal{S}}$ occurs as the image of a toric prequotient of an open toric subvariety of some $\mathbb{C}^s$. Our construction is the following: For every $i \in I$, let R_i denote the set of all pairs (ϱ, i), where $\varrho \in \Delta_{ii}^{(1)}$. Note that $R_i \subset \mathfrak{F}(\mathcal{S})$. Set

$$N' := \bigoplus_{i \in I} \mathbb{Z}^{R_i}, \qquad \widetilde{N} := N \oplus N'.$$

For every ray $\varrho \in \bigcup_{i \in I} \Delta_{ii}^{(1)}$ let $v_\varrho \in N$ denote the primitive lattice vector contained in ϱ. Define lattice homomorphisms

$$Q' \colon N' \to N, \quad Q'(e_{(\varrho,i)}) := v_\varrho, \qquad Q := \operatorname{id}_N + Q' \colon \widetilde{N} \to N.$$

Here the $e_{(\varrho,i)}$, $\varrho \in \bigcup_{i \in I} \Delta_{ii}^{(1)}$, denote the canonical basis vectors of $\mathbb{Z}^{R_i}$. Now, choose an ordering "$\le$" of I and define an index set $\widetilde{I}$ by

$$\widetilde{I} := \{(\tau, i, j);\ i \le j \in I,\ \tau \in \Delta_{ij}^{\max}\}.$$

So, as a set, $\widetilde{I}$ is isomorphic to the disjoint union of all $\Delta_{ij}^{\max}$, $i \le j$. For every index $k = (\tau, i, j) \in \widetilde{I}$ we define a strictly convex lattice cone $\widetilde{\sigma}_k$ in $N' \subset \widetilde{N}$ by setting

$$\widetilde{\sigma}_k := \operatorname{cone}(e_{(\varrho,l)};\ l \in \{i,j\},\ \varrho \prec \tau).$$

In particular, if $k = (\sigma, i, i)$ with the maximal cone $\sigma \in \Delta_{ii}$, then $\widetilde{\sigma}_k$ is the positive quadrant $\mathbb{R}^{R_i}_{\ge 0}$. By construction, the cones $\widetilde{\sigma}_k$ are the maximal cones of a fan $\widetilde{\Delta}$ in $\widetilde{N}$, and $X_{\widetilde{\Delta}}$ is isomorphic to an open toric subvariety of $\mathbb{C}^s$, where $s := \dim(\widetilde{N})$.

Now let $\widetilde{\mathcal{S}}$ denote the affine system of fans determined by $\widetilde{\Delta}$, i.e., $\widetilde{\mathcal{S}} = (\widetilde{\Delta}_{kk'})_{k,k' \in \widetilde{I}}$, where $\widetilde{\Delta}_{kk'}$ is the fan of faces of $\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'}$.

8.1 Lemma. *For any two elements $k = (\tau, i, j)$ and $k' = (\tau', i', j')$ of $\widetilde{I}$ we have*

i) $Q_{\mathbb{R}}(\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'}) = \{0\} \in \Delta_{ii'}$, *if* $\{i, j\} \cap \{i', j'\} = \emptyset$.

ii) $Q_{\mathbb{R}}(\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'}) = \tau \cap \tau' \in \Delta_{ii'}$, *if* $\{i, j\} \cap \{i', j'\} \neq \emptyset$.

Proof. Note first that by definition the intersection of $\widetilde{\sigma}_k$ and $\widetilde{\sigma}_{k'}$ is given by

$$\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'} = \mathrm{cone}(e_{(\varrho,l)};\ l \in \{i,j\} \cap \{i',j'\},\ \varrho \prec \tau,\ \varrho \prec \tau').$$

If $\{i,j\} \cap \{i',j'\}$ is empty, then $Q_{\mathbb{R}}(\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'}) = \{0\} \in \Delta_{ii'}$. So assume that $\{i,j\} \cap \{i',j'\}$ is not empty. As an example we treat the case $j = j'$. Then $\tau, \tau' \in \Delta_{jj}$, in particular, $\tau \cap \tau'$ is a face of both, τ and τ'. Thus we obtain

$$Q_{\mathbb{R}}(\widetilde{\sigma}_k \cap \widetilde{\sigma}_{k'}) = \mathrm{cone}(v_\varrho;\ \varrho \prec \tau \cap \tau') = \tau \cap \tau' \in \Delta_{ij} \cap \Delta_{i'j} \subset \Delta_{ii'}. \quad \square$$

By the above lemma, the map $\mu: \widetilde{I} \to I$, $(\tau, i, j) \mapsto i$, satisfies the condition $(*)$ of Lemma 5.6, and hence defines a map $(Q, \mathfrak{q})$ of the systems of fans $\widetilde{\mathcal{S}}$ and $\mathcal{S}$. Let H denote the subtorus of the acting torus $\widetilde{T}$ of $X_{\widetilde{\mathcal{S}}} = X_{\widetilde{\Delta}}$ that corresponds to the primitive sublattice $\ker(Q) \subset \widetilde{N}$. Then we obtain:

8.2 Proposition. *The toric morphism $q: X_{\widetilde{\mathcal{S}}} \to X_{\mathcal{S}}$ associated to $(Q, \mathfrak{q})$ is the toric prequotient for the action of H on $X_{\widetilde{\mathcal{S}}}$. Moreover, q is even a categorical prequotient for the H–action.*

Proof. It is clear that q is a surjective toric prequotient. To see that it is categorical, we have to show the existence of factorizations. So, let $f: X_{\widetilde{\mathcal{S}}} \to Z$ be an H-invariant regular map. As usual, for $i \in I$, let X_i denote the chart $X_{\Delta_{ii}}$ in $X_{\mathcal{S}}$. For every $k = (\tau, i, j) \in \widetilde{I}$ the lattice homomorphism Q gives rise to a toric morphism

$$q_k \ = \ q|_{X_{\widetilde{\sigma}_k}}: X_{\widetilde{\sigma}_k} \to X_\tau \ \subset \ X_i \cap X_j \ \subset \ X_{\mathcal{S}}.$$

Note that the q_k are algebraic quotients for the action of H on the $X_{\widetilde{\sigma}_k}$. In particular, since algebraic quotients are categorical prequotients (see Proposition 6.4), we obtain for every $k = (\tau, i, j) \in \widetilde{I}$ a regular map $\widetilde{f}_k: X_\tau \to Z$ such that

$$f|_{X_{\widetilde{\sigma}_k}} = \widetilde{f}_k \circ q_k.$$

We claim that the $\widetilde{f}_k$ glue together to a map $\widetilde{f}: X_{\mathcal{S}} \to Z$. To see this, consider first $k = (\tau, i, j)$ and $k' = (\sigma_i, i, i)$, where σ_i denotes the maximal cone in Δ_{ii}. Recall that

$$\widetilde{\sigma}_{k'} \cap \widetilde{\sigma}_k = \mathrm{cone}(e_{(\rho,i)};\ \varrho \prec \tau) =: \widetilde{\tau}_i.$$

In particular, we have $Q_{\mathbb{R}}(\widetilde{\tau}_i) = \tau$. Consequently, the restriction of q_k to $X_{\widetilde{\tau}_i}$ maps $X_{\widetilde{\tau}_i}$ onto X_τ and is again an algebraic quotient. Since $\widetilde{f}_{k'} \circ q_k$ coincides with f on $X_{\widetilde{\tau}_i}$, we can conclude that

$$\widetilde{f}_{k'}|_{X_\tau} = \widetilde{f}_k.$$

Thus, to obtain the claim, only the cases $k = (\sigma_i, i, i)$ and $k' = (\sigma_j, j, j)$ remain to be treated. We have to consider

$$X_i \cap X_j = \bigcup_{\tau \in \Delta_{ij}^{\max}} X_\tau.$$

For every $\tau \in \Delta_{ij}^{\max}$ the previous consideration yields $\widetilde{f}_k|_{X_\tau} = \widetilde{f}_{(\tau,i,j)} = \widetilde{f}_{k'}|_{X_\tau}$. Therefore $\widetilde{f}_k$ and $\widetilde{f}_{k'}$ in fact coincide on $X_i \cap X_j$, which proves our claim. By construction we have $f = \widetilde{f} \circ q$. $\square$

8.3 Corollary. *Every toric prevariety X occurs as the image of a categorical prequotient of an open toric subvariety of some $\mathbb{C}^s$.* □

In general, the toric prequotient q constructed above, is not a good prequotient. In the remaining part of this section we will investigate when a given toric prevariety $X_{\mathcal{S}}$ can be obtained as a good prequotient of an open toric subvariety of some $\mathbb{C}^s$.

Call a toric prevariety X with acting torus T of *affine intersection* if for any two maximal affine T–stable charts $X_1, X_2 \subset X$ their intersection $X_1 \cap X_2$ is again affine. Note that every toric variety is of affine intersection while for toric prevarieties this a proper condition, as the following example shows:

8.4 Example. Let $\sigma := \operatorname{cone}(e_1, e_2) \subset \mathbb{R}^2$ and let $\mathcal{S}$ denote the system of fans with $\Delta_{11} = \Delta_{22} = \mathfrak{F}(\sigma)$ and

$$\Delta_{12} = \Delta_{21} = \{\{0\}, \mathbb{R}_{\geq 0} e_1, \mathbb{R}_{\geq 0} e_2\}.$$

Then the toric prevariety $X_{\mathcal{S}}$ is just $\mathbb{C}^2$ with doubled zero. Clearly $X_{\mathcal{S}}$ is not of affine intersection. ◇

8.5 Proposition. *If there is a good prequotient $q\colon \widetilde{X} \to X_{\mathcal{S}}$ with a toric variety $\widetilde{X}$, then $X_{\mathcal{S}}$ is of affine intersection.*

Proof. Let X_i, $i = 1, \ldots, r$, be the maximal T-stable affine open subsets of $X_{\mathcal{S}}$ and set $\widetilde{X}_i := q^{-1}(X_i)$. Then the restrictions $q_{ij}\colon \widetilde{X}_i \cap \widetilde{X}_j \to X_i \cap X_j$ of q are good prequotients. Since $\widetilde{X}_i$ and $\widetilde{X}_j$ are affine, so is $\widetilde{X}_i \cap \widetilde{X}_j$ and consequently $X_i \cap X_j$. □

In the sequel assume that $X_{\mathcal{S}}$ is of affine intersection. Since $\mathcal{S}$ was assumed to be affine this means that for any two $i, j \in I$ the set $\Delta_{ij}^{\max}$ consists of a single cone σ_{ij}. We show that $X_{\mathcal{S}}$ occurs as the image of a good prequotient of an open toric subvariety of some $\mathbb{C}^s$ using the following generalization of Cox's construction (see [5]):

Let R denote the set of equivalence classes $[\varrho, i] \in \Omega(\mathcal{S})$ where ϱ is one–dimensional. Set $N' := \mathbb{Z}^R$ and $\widetilde{N} := N \oplus N'$. As before, denote for $\varrho \in \bigcup \Delta_{ij}^{(1)}$ by v_ϱ the primitive lattice vector contained in ϱ. Define lattice homomorphisms

$$Q'\colon N' \to N, \quad Q(e_{[\varrho,i]}) := v_\varrho, \qquad Q := \operatorname{id}_N + Q'\colon \widetilde{N} \to N.$$

For every $i \in I$ define a strictly convex cone in $\widetilde{N}$ by setting

$$\widetilde{\sigma}_i := \operatorname{cone}(e_{[\varrho,i]};\ \varrho \in \Delta_{ii}^{(1)})$$

Then the cones $\widetilde{\sigma}_i$, $i \in I$, are the maximal cones of a fan $\widetilde{\Delta}$ in $\widetilde{N}$. Let $\widetilde{\mathcal{S}}$ denote the affine system of fans associated to $\widetilde{\Delta}$.

8.6 Lemma. *The homomorphism Q together with $\mu := \operatorname{id}_I$ determines a map of systems of fans $(Q, \mathfrak{q})$ from $\widetilde{\mathcal{S}}$ to $\mathcal{S}$.*

Proof. We have to verify condition $(*)$ of Lemma 5.6 for μ. Note first that for $i, j \in I$ we have

$$\widetilde{\sigma}_i \cap \widetilde{\sigma}_k = \operatorname{cone}(e_{[\varrho,i]};\ \varrho \in \Delta_{ij}^{(1)}).$$

Moreover, since $\mathcal{S}$ is affine and $X_{\mathcal{S}}$ is of affine intersection, Δ_{ij} is the fan of faces of a single cone σ_{ij}. Hence one obtains condition $(*)$ of Lemma 5.6 for μ from

$$Q_{\mathbb{R}}(\widetilde{\sigma}_i \cap \widetilde{\sigma}_k) = \operatorname{cone}(v_\varrho; \varrho \in \Delta_{ij}^{(1)}) = \sigma_{ij} \in \Delta_{ij}. \qquad \square$$

By construction, the toric morphism $q\colon X_{\widetilde{\mathcal{S}}} \to X_{\mathcal{S}}$ defined by $(Q, \mathfrak{q})$ is surjective and affine. Thus, denoting by H the kernel of the homomrphism of acting tori associated to q, we infer from Corollary 6.9:

8.7 Proposition. *The toric morphism* $q: X_{\widetilde{\mathcal{S}}} \to X_{\mathcal{S}}$ *is a good prequotient for the action of* H *on* $X_{\widetilde{\mathcal{S}}}$. □

Together with Proposition 8.5, the above proposition yields the following

8.8 Theorem. *For any toric prevariety* X*, the following statements are equivalent:*

i) *There is an open toric subvariety* U *of some* $\mathbb{C}^n$ *and a good prequotient* $q: U \to X$.

ii) X *is of affine intersection.* □

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